ECE 346: Digital Signal Processing — Spring 2021
Soo Min Kwon — Week of March 22, 2021

Recitation #7: Z-Transform, IIR and FIR Filters

Objective & Outline

- Problems $1-4$: recitation problems
- Problem 5: self-assessment problem

Welcome back! I hope everyone had a nice Spring break and is now ready to tackle the rest of this semester. Below are a few random important points on FIR/IIR filters in case you forgot:

- In the real world, we cannot always implement ideal filters. Instead, we can approximate an ideal filter by an infinite impulse response (IIR) filter that can be described by a **difference equation**.
- A **causal** filter is one in which $h[n] = 0$ for all $n < 0$.
- An LTI system can be expressed a constant coefficient difference equation (CCDE) whenever the z-transform of that impulse response can be expressed as a rational function to two polynomials in z^{-1}:

$$H(z) = \frac{P(z)}{Q(z)} = \frac{p_0 + p_1 z^{-1} + \ldots + p_M z^{-M}}{q_0 + q_1 z^{-1} + \ldots + q_M z^{-M}} \tag{1}$$

- The z-transform of an FIR filter is simply a sum of finite terms.
- A **casual FIR filter** only has poles at $z = 0$ and $z = \infty$.
- In an FIR filter, the location of the zeros help up interpret the frequency response of the filter.
  - The closer to zero is to the unit-circle, the more attenuated our frequencies are in the direction of the zero.

Note that you can also find a short note on the region of convergence (ROC) of z-transforms on Canvas. The problems start on the following page.

Problem 1 (Z-Transforms)**.** Recall that the z-transform "analysis" equation is given by

$$X(z) = \sum_{n=\infty}^{\infty} x[n]z^{-n}. \tag{2}$$

Determine the z-transform for each of the following sequences:

(a) $x_1[n] = \frac{1}{2^n}u[n]$

(b) $x_2[n] = -\frac{1}{2^n}u[-n-1]$

(c) $x_3[n] = \frac{1}{3^n}u[n-2]$

Problem 1 :-

(a) $x_1[n] = \frac{1}{2^n} u[n]$

By applying the z-transform equation:

$$X_1(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2^n} \underbrace{u[n]}_{1 \text{ for all } n \geq 0} z^{-n}.$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n$$

this is because we have a right-sided signal due to u[n].

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - 1/2}, \quad |z| > 1/2.$$

(b) $x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1].$

$$= -\left(\frac{1}{2}\right)^n u[-(n+1)]$$

Again, by applying the equation:

$$X_2(z) = \sum_{n=-\infty}^{\infty} -\left(\frac{1}{2}\right)^n u[-(n+1)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -\left(\frac{1}{2}\right)^n z^{-n}.$$

$$= \sum_{n=1}^{\infty} -\left(\frac{1}{2}\right)^{-n} z^{n}$$

$$= 1 + \sum_{n=0}^{\infty} -(2)^n z^n$$

$$= 1 - \sum_{n=0}^{\infty} (2z)^n$$

$$= 1 - \frac{1}{1-2z} = \frac{1-2z}{1-2z} - \frac{1}{1-2z}$$

$$= \frac{-2z}{1-2z} = \frac{z}{z - 1/2}$$

$$X_2(z) = \frac{z}{z - 1/2} \quad |z| < 1/2.$$

due to u[-n-1].

(c) $x_3[n] = \left(\frac{1}{3}\right)^n u[n-2]$

Applying the definition:

$$X_3(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n-2]\, z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\tfrac{1}{3}\right)^n z^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{3z}\right)^n$$

$$= -1 - \frac{1}{3} z^{-1} + \sum_{n=0}^{\infty} \left(\frac{2}{3z}\right)^n$$

$$+ \frac{z}{z - 1/3}$$

$$= \frac{-\left(1 + 1/3\, z^{-1}\right)\left(z - 1/3\right) + z}{z - 1/3}$$

$$= \frac{-\cancel{z} + \cancel{1/3} - \cancel{1/3} + 1/3\, z^{-1} + \cancel{z}}{z - 1/3}$$

$$= \frac{1/9\, z^{-1}}{z - 1/3} = \frac{1/9}{z(z - 1/3)}.$$

What are the poles?

$$z(z - 1/3) = 0 \Rightarrow \text{poles} = \{0, 1/3\}$$

And since we have $u[n-2]$, the ROC is

$$|z| > 1/3.$$

Thus,

$$X_3(z) = \frac{1/9}{z(z - 1/3)}, \quad |z| > 1/3.$$

∎

Problem 2 (A Harder Z-Transform Problem)**.** When the input to an LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1], \tag{3}$$

the output is

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]. \tag{4}$$

(a) Find the z-transforms of $x[n]$ and $y[n]$. Write them as ratios of factorized polynomials in z^{-1}.

(b) Find the system transfer function $H(z)$.

(c) Plot the poles and zeros of this function and indicate the region of convergence (ROC).

(d) Is the transfer function $H(z)$ stable? Is it causal? Explain why or why not.

Problem 2:-

(a) $x[n] = \underbrace{\left(\frac{1}{2}\right)^n u[n]}_{x_1[n]} + \underbrace{2^n u[-n-1]}_{x_2[n]}$

$= \mathcal{Z}\{x_1[n]\} + \mathcal{Z}\{x_2[n]\}$, where

$= X_1(z) + X_2(z)$ $\rightsquigarrow$ linearity of z-transform

$$X_1(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]\, z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} 2^n u[-(n+1)]\, z^{-n}$$
$$= \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{n=1}^{\infty} 2^{-n} z^{n} = \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n$$
$$= -1 + \frac{1}{1-z/2}$$
$$= \frac{-1-z/2}{1-z/2} + \frac{1}{1-z/2}$$
$$= \frac{-z/2}{1-z/2} = \frac{-z}{2-z}$$
$$= \frac{-1}{1-2z^{-1}}.$$

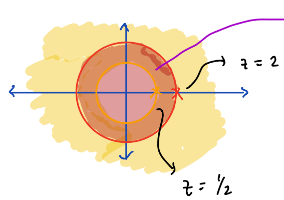
$$X(z) = X_1(z) + X_2(z)$$
$$= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}.$$

The poles of $X(z)$ are

poles $= \{ 1/2 , 2 \}$

right-sided   left-sided

and so the ROC will look like



$\frac{1}{2} < |z| < 2$.

$$y[n] = \underbrace{6\left(\tfrac{1}{2}\right)^n u[n]}_{y_1[n]} - \underbrace{6\left(\tfrac{3}{4}\right)^n u[n]}_{y_2[n]}$$

$$\begin{aligned} Y_1(z) &= \sum_{n=-\infty}^{\infty} y_1[n] z^{-n} \\ &= \sum_{n=0}^{\infty} 6\left(\tfrac{1}{2}\right) z^{-n} \\ &= 6 \sum_{n=0}^{\infty} \left(\tfrac{1}{2z}\right) \\ &= \frac{6}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

$$\begin{aligned} Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2[n] z^{-n} \\ &= \sum_{n=0}^{\infty} 6\left(\tfrac{3}{4}\right)^n z^{-n} \\ &= 6 \sum_{n=0}^{\infty} \left(\tfrac{3}{4z}\right)^n \\ &= \frac{6}{1-\frac{3}{4}z^{-1}} \end{aligned}$$

$$\begin{aligned} Y(z) &= Y_1(z) + Y_2(z) \\ &= \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} \end{aligned}$$

The poles of this system are

$$\text{poles} = \{ 1/2,\ 3/4 \}.$$

Since they are both right-sided, the overlapping region would be $|z| > 3/4$, and so

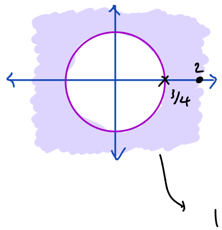
$$Y(z) = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}, \quad |z| > 3/4.$$

(b) $$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\dfrac{-\frac{3}{\cancel{2}}\cancel{z^{-1}}}{\left(1-\cancel{\frac{1}{2}z^{-1}}\right)\left(1-\frac{3}{4}z^{-1}\right)}}{\dfrac{-3/\cancel{2}\, z^{-1}}{\left(1-\cancel{1/2\, z^{-1}}\right)\left(1-2z^{-1}\right)}.} = \frac{1-2z^{-1}}{1-3/4\, z^{-1}}.$$

(c) $$H(z) = \frac{1-2z^{-1}}{1-3/4\, z^{-1}} = \frac{z-2}{z-3/4}$$

$$\text{poles} = \{ 3/4 \}$$
$$\text{zeros} = \{ 2 \}$$



For the ROC, we have a single pole at 3/4 and a zero at 2. Since the ROC for $Y(z)$ is the intersection of the ROCs for $X(z)$ and $H(z)$, we have $|z| > 3/4$ for the ROC of $H(z)$ as well.

(d) Since the transfer function has all poles inside the unit circle and the ROC extends from the largest pole to ∞, the system is causal and stable.

If you have a hard time understanding this, refer to the PDF notes on ROCs.

Problem 3 (FIR and IIR Systems)**.** Consider a DT LTI system with transfer function defined by

$$H(z) = \frac{0.75z^{-2} + z^{-1} - 1}{1 - 2z^{-1}} \tag{5}$$

(a) Is this system an FIR or an IIR system?

(b) Draw a pole-zero diagram for this system.

(c) Specify the ROC of this system under the assumption that the DTFT of its impulse response exists.

(d) Mathematically express the DTFT of this system.

Problem 3:-

(a) Since $H(z)$ is written as a rational function of two polynomials in the form

$$H(z) = \frac{P(z)}{Q(z)},$$

this system is an IIR filter.

(b) $$H(z) = \frac{0.75z^{-2} + z^{-1} - 1}{1 - 2z^{-1}} = \frac{0.75 + z - z^2}{z^2 - 2}$$

We can find the zeroes of $H(z)$ by using the quadratic formula, where

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

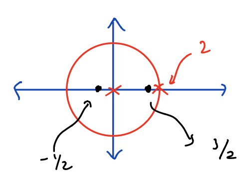
where $a = -1$, $b = 1$, $c = 0.75$.

$$\text{zeroes}: \{-1/2, 3/2\}$$

For the poles:

$$z(z-2) = 0$$

$$\text{poles}: \{0, 2\}.$$



(c) Since the DTFT of $h[n]$ exists, the ROC must contain the unit circle. Also, since the ROC cannot contain the poles,

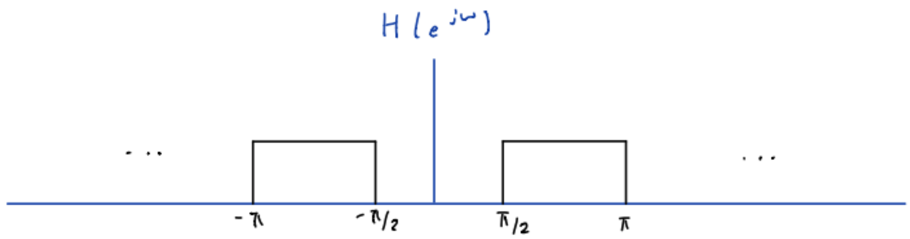
$$0 < |z| < 2.$$

(d) Using the relationship $z = e^{j\omega}$:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{0.75e^{-j2\omega} + e^{-j\omega} - 1}{1 - 2e^{-j\omega}}.$$

■

Problem 4 (FIR and IIR Systems)**.** Consider a DT LTI system that is an ideal highpass filter with cutoff frequencies $\frac{\pi}{2}$ and π:



(a) Is this system an FIR or an IIR system?

(b) Find the impulse response of this filter, $h[n]$.

(c) Is this filter a causal or non-causal system?

Problem 4:-

(a) The intuition for this is the following:

$H(e^{j\omega})$ is finite in frequency from $[-\pi, \pi]$

$\Downarrow$

$h[n]$ must be infinite in time $(\text{sinc}(\cdot))$

$\Downarrow$

$h[n]$ has an infinite impulse response

$\Downarrow$

IIR filter.

(b) We have seen this before:

$$h[n] = \frac{\sin(n\pi)}{\pi n} - \frac{\sin(\pi/2\, n)}{\pi n}.$$

(c) A system is causal if

$$h[n] = 0, \quad \forall n < 0.$$

Since $h[n]$ clearly does not satisfy this criteria, $h[n]$ is anti-causal.

$\boxtimes$

Problem 5 (Self-assessment)**.** As usual, try to work on these problems together in break-out rooms.

1. Consider an FIR filter with impulse response

$$h[n] = \begin{cases} \frac{1}{3}, & n = 0, 1, \ldots, 5 \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

   How many addition and multiplication *signs* for each n are needed in order to produce a filtered output $y[n] = x[n] * h[n]$ for an arbitrary input $x[n]$?

2. Suppose that the transfer function of a filter is given by

$$H(z) = \frac{1 - z^{-2}}{1 - 2z^{-4}}. \tag{7}$$

   (a) How many poles and zeros does this transfer function $H(z)$ have?

   (b) Suppose that the DTFT of $h[n]$ exists. What does this mean in terms of right, left, or a two-sided sequence?

Problem 5:-

#1:

(a) Using the definition of discrete-time convolution:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m]$$
$$= \sum_{m=0}^{5} \frac{1}{3} x[n-m] .$$

If you count the signs explicitly, then there are

6 multiplications & 5 additions.

#2:

(a) $$H(z) = \frac{1 - z^{-2}}{1 - 2z^{-4}} = \frac{z^4 - z^2}{z^4 - 2}$$

Solving for zeroes, we get

$$z^2 (z^2 - 1) = 0 .$$

order of 2.

And thus,

$$\text{zeroes} = \{ 0, -1, 1 \}$$

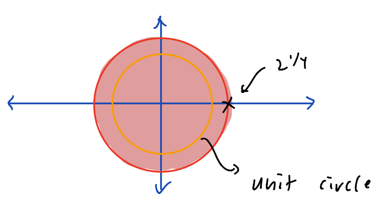
For the poles, we have

$$\text{poles} : \{ 2^{1/4}, -2^{1/4}, j2^{1/4}, -j2^{1/4} \}.$$

Note that we need (and did) to satisfy

\# of poles = \# of zeroes.

(b) Since the DTFT exists, our ROC must contain the unit circle, and so our ROC must be left-sided:



$$|z| < 2^{1/4} .$$